# Lecture 02: Mathematical Basics (Probability)

# Probability Basics

- Sample Space: $\Omega$ is a set of outcomes (it can either be finite or infinite)
- Random Variable: $\mathbb{X}$ is a random variable that assigns probabilities to outcomes

Example: Let $\Omega = \{\text{Heads}, \text{Tails}\}$. Let $\mathbb{X}$ be a random variable that outputs Heads with probability $1/3$ and outputs Tails with probability $2/3$

- The probability that $\mathbb{X}$ assigns to the outcome $x$ is represented by

$$\mathbb{P}\left[\mathbb{X} = x\right]$$

Example: In the ongoing example $\mathbb{P}\left[\mathbb{X} = \text{Heads}\right] = 1/3$.

# Function of a Random Variable

- Let $f \colon \Omega \to \Omega'$ be a function
- Let $\mathbb{X}$ be a random variable over the sample space $\mathbb{X}$
- We define a new random variable $f(\mathbb{X})$ is over $\Omega'$ as follows

$$\mathbb{P}\left[f(\mathbb{X}) = y\right] = \sum_{x \in \Omega \colon f(x) = y} \mathbb{P}\left[\mathbb{X} = x\right]$$

# Joint Distribution and Marginal Distributions I

- Suppose $(\mathbb{X}_1, \mathbb{X}_2)$ is a random variable over $\Omega_1 \times \Omega_2$.
  - Intuitively, the random variable $(\mathbb{X}_1, \mathbb{X}_2)$ takes values of the form $(x_1, x_2)$, where the first coordinate lies in $\Omega_1$, and the second coordinate lies in $\Omega_2$

For example, let $(\mathbb{X}_1, \mathbb{X}_2)$ represent the temperatures of West Lafayette and Lafayette. Their sample space is $\mathbb{Z} \times \mathbb{Z}$. Note that these two outcomes can be correlated with each other.

# Joint Distribution and Marginal Distributions II

- Let $P_1 : \Omega_1 \times \Omega_2 \to \Omega_1$ be the function $P_1(x_1, x_2) = x_1$ (the projection operator)
- So, the random variable $P_1(\mathbb{X}_1, \mathbb{X}_2)$ is a probability distribution over the sample space $\Omega_1$
- This is represented simply as $\mathbb{X}_1$, the marginal distribution of the first coordinate
- Similarly, we can define $\mathbb{X}_2$

# Conditional Distribution

- Let $(\mathbb{X}_1, \mathbb{X}_2)$ be a joint distribution over the sample space $\Omega_1 \times \Omega_2$
- We can define the distribution $(\mathbb{X}_1 \mid \mathbb{X}_2 = x_2)$ as follows
  - This random variable is a distribution over the sample space $\Omega_1$
  - The probability distribution is defined as follows

$$\mathbb{P}\left[\mathbb{X}_1 = x_1 \mid \mathbb{X}_2 = x_2\right] = \frac{\mathbb{P}\left[\mathbb{X}_1 = x_1, \mathbb{X}_2 = x_2\right]}{\sum_{x \in \Omega_1} \mathbb{P}\left[\mathbb{X}_1 = x, \mathbb{X}_2 = x_2\right]}$$

For example, conditioned on the temperature at Lafayette being 0, what is the conditional probability distribution of the temperature in West Lafayette?

**Theorem (Bayes' Rule)**

*Let $(\mathbb{X}_1, \mathbb{X}_2)$ be a joint distribution over the sample space $(\Omega_1, \Omega_2)$. Let $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$ be such that $\mathbb{P}[\mathbb{X}_1 = x_1, \mathbb{X}_2 = x_2] > 0$. Then, the following holds.*

$$\mathbb{P}\left[\mathbb{X}_1 = x_1 \mid \mathbb{X}_2 = x_2\right] = \frac{\mathbb{P}[\mathbb{X}_1 = x_1, \mathbb{X}_2 = x_2]}{\mathbb{P}[\mathbb{X}_2 = x_2]}$$

The random variables $\mathbb{X}_1$ and $\mathbb{X}_2$ are independent of each other if the distribution $(\mathbb{X}_1 \mid \mathbb{X}_2 = x_2)$ is identical to the random variable $\mathbb{X}_1$, for all $x_2 \in \Omega_2$ such that $\mathbb{P}[\mathbb{X}_2 = x_2] > 0$

We can generalize the Bayes' Rule as follows.

**Theorem (Chain Rule)**

*Let* $(\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_n)$ *be a joint distribution over the sample space* $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$*. For any* $(x_1, \ldots, x_n) \in \Omega_1 \times \cdots \times \Omega_n$ *we have*

$$\mathbb{P}[\mathbb{X}_1 = x_1, \ldots, \mathbb{X}_n = x_n] = \prod_{i=1}^{n} \mathbb{P}\left[\mathbb{X}_i = x_i \mid \mathbb{X}_{i-1} = x_{i-1} \ldots, \mathbb{X}_1 = x_1\right]$$

# Important: Why use Bayes' Rule I

In which context do we foresee to use the Bayes' Rule to compute joint probability?

- Sometimes, the problem at hand will clearly state how to sample $\mathbb{X}_1$ and then, conditioned on the fact that $\mathbb{X}_1 = x_1$, it will state how to sample $\mathbb{X}_2$. In such cases, we shall use the Bayes' rule to calculate

$$\mathbb{P}\left[\mathbb{X}_1 = x_1, \mathbb{X}_2 = x_2\right] = \mathbb{P}\left[\mathbb{X}_1 = x_1\right] \mathbb{P}\left[\mathbb{X}_2 = x_2 | \mathbb{X}_1 = x_1\right]$$

- Let us consider an example.
    - Suppose $\mathbb{X}_1$ is a random variable over $\Omega_1 = \{0, 1\}$ such that $\mathbb{P}\left[X_1 = 0\right] = 1/2$. Next, the random variable $\mathbb{X}_2$ is over $\Omega_2 = \{0, 1\}$ such that $\mathbb{P}\left[X_2 = x_1 | \mathbb{X}_1 = x_1\right] = 2/3$. Note that $\mathbb{X}_2$ is biased towards the outcome of $\mathbb{X}_1$.
    - What is the probability that we get $\mathbb{P}\left[\mathbb{X}_1 = 0, \mathbb{X}_2 = 1\right]$?

# Important: Why use Bayes' Rule II

- To compute this probability, we shall use the Bayes' rule.

$$\mathbb{P}\left[\mathbb{X}_1 = 0\right] = 1/2$$

Next, we know that

$$\mathbb{P}\left[\mathbb{X}_2 = 0 | \mathbb{X}_1 = 0\right] = 2/3$$

Therefore, we have $\mathbb{P}\left[\mathbb{X}_2 = 1 | \mathbb{X}_1 = 0\right] = 1/3$. So, we get

$$\begin{aligned}\mathbb{P}\left[\mathbb{X}_1 = 0, \mathbb{X}_2 = 1\right] &= \mathbb{P}\left[\mathbb{X}_1 = 0\right] \mathbb{P}\left[\mathbb{X}_2 = 1 | \mathbb{X}_1 = 0\right] \\ &= (1/2) \cdot (1/3) = 1/6\end{aligned}$$

# Independence of Random Variables

- Consider a joint distribution $(\mathbb{X}_1, \mathbb{X}_2)$ over the sample space $\Omega_1 \times \Omega_2$
- The marginal distributions $\mathbb{X}_1$ and $\mathbb{X}_2$ are independent of each other, if for all $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$ we have: If $\mathbb{P}\left[\mathbb{X}_2 = x_2\right] > 0$ then

$$\mathbb{P}\left[\mathbb{X}_1 = x_1\right] = \mathbb{P}\left[\mathbb{X}_1 = x_1 | \mathbb{X}_2 = x_2\right].$$

- Equivalently, the following condition is satisfied

$$\mathbb{P}\left[\mathbb{X}_1 = x_1\right] \cdot \mathbb{P}\left[\mathbb{X}_2 = x_2\right] = \mathbb{P}\left[\mathbb{X}_1 = x_1, \mathbb{X}_2 = x_2\right].$$

- Let $\mathbb{S}$ be the random variable representing whether I studied for my exam. This random variable has sample space $\Omega_1 = \{\mathsf{Y}, \mathsf{N}\}$
- Let $\mathbb{P}$ be the random variable representing whether I passed my exam This random variable has sample space $\Omega_2 = \{\mathsf{Y}, \mathsf{N}\}$
- Our sample space is $\Omega = \Omega_1 \times \Omega_2$
- The joint distribution $(\mathbb{S}, \mathbb{P})$ is represented in the next page

# Probability: First Example II

| $s$ | $p$ | $\mathbb{P}[\mathbb{S}=s,\mathbb{P}=p]$ |
|---|---|---|
| Y | Y | 1/2 |
| Y | N | 1/4 |
| N | Y | 0 |
| N | N | 1/4 |

# Probability: First Example III

Here are some interesting probability computations
The probability that I pass.

$$\mathbb{P}\left[\mathbb{P} = \mathsf{Y}\right] = \mathbb{P}\left[\mathbb{S} = \mathsf{Y}, \mathbb{P} = \mathsf{Y}\right] + \mathbb{P}\left[\mathbb{S} = \mathsf{N}, \mathbb{P} = \mathsf{Y}\right]$$
$$= 1/2 + 0 = 1/2$$

# Probability: First Example IV

The probability that I study.

$$\begin{aligned}\mathbb{P}[\mathbb{S} = \mathsf{Y}] &= \mathbb{P}[\mathbb{S} = \mathsf{Y}, \mathbb{P} = \mathsf{Y}] + \mathbb{P}[\mathbb{S} = \mathsf{Y}, \mathbb{P} = \mathsf{N}] \\ &= 1/2 + 1/4 = 3/4\end{aligned}$$

# Probability: First Example V

The probability that I pass conditioned on the fact that I studied.

$$\mathbb{P}\left[\mathbb{P} = \mathsf{Y} \mid \mathbb{S} = \mathsf{Y}\right] = \frac{\mathbb{P}\left[\mathbb{P} = \mathsf{Y}, \mathbb{S} = \mathsf{Y}\right]}{\mathbb{P}\left[\mathbb{S} = \mathsf{Y}\right]}$$

$$= \frac{1/2}{3/4} = \frac{2}{3}$$

# Probability: Second Example I

- Let $\mathbb{T}$ be the time of the day that I wake up. The random variable $\mathbb{T}$ has sample space $\Omega_1 = \{4, 5, 6, 7, 8, 9, 10\}$
- Let $\mathbb{B}$ represent whether I have breakfast or not. The random variable $\mathbb{B}$ has sample space $\Omega_2 = \{T, F\}$
- Our sample space is $\Omega = \Omega_1 \times \Omega_2$
- The joint distribution of $(\mathbb{T}, \mathbb{B})$ is presented on the next page

# Probability: Second Example II

| $t$ | $b$ | $\mathbb{P}[\mathbb{T}=t,\mathbb{B}=b]$ |
|---|---|---|
| 4 | T | 0.03 |
| 4 | F | 0 |
| 5 | T | 0.02 |
| 5 | F | 0 |
| 6 | T | 0.30 |
| 6 | F | 0.05 |
| 7 | T | 0.20 |
| 7 | F | 0.10 |
| 8 | T | 0.10 |
| 8 | F | 0.08 |
| 9 | T | 0.05 |
| 9 | F | 0.05 |
| 10 | T | 0 |
| 10 | F | 0.02 |

# Probability: Second Example III

- What is the probability that I have breakfast conditioned on the fact that I wake up at or before 7?

Formally, what is $\mathbb{P}\left[\mathbb{B} = \mathsf{T} \mid \mathbb{T} \leqslant 7\right]$?